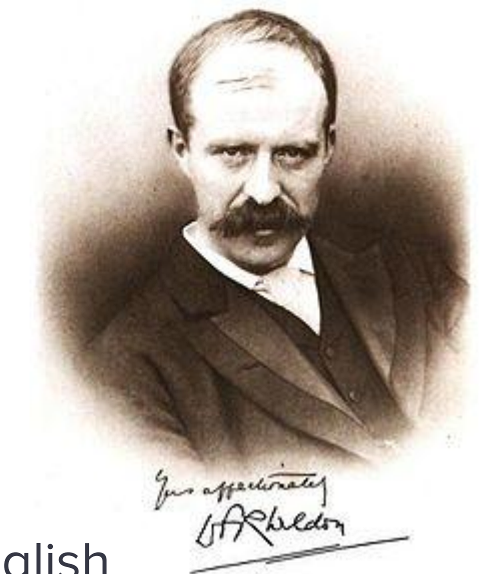
Section 6.3
Chi-Square test of GOF

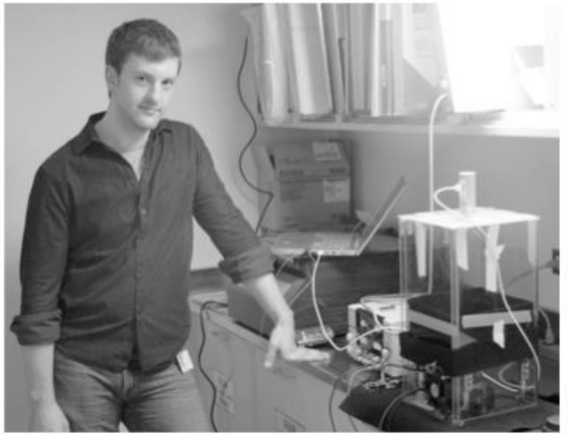
Stats 7 Summer Session II 2022

Weldon's dice



- Walter Frank Raphael Weldon (1860 - 1906), was an English evolutionary biologist and a founder of biometry. He was the joint founding editor of Biometrika, with Francis Galton and Karl Pearson.
- In 1894, he rolled 12 dice 26,306 times, and recorded the number of 5s or 6s (which he considered to be a success).
- It was observed that 5s or 6s occurred more often than expected, and Pearson hypothesized that this was probably due to the construction of the dice. Most inexpensive dice have hollowed-out pips, and since opposite sides add to 7, the face with 6 pips is lighter than its opposing face, which has only 1 pip.

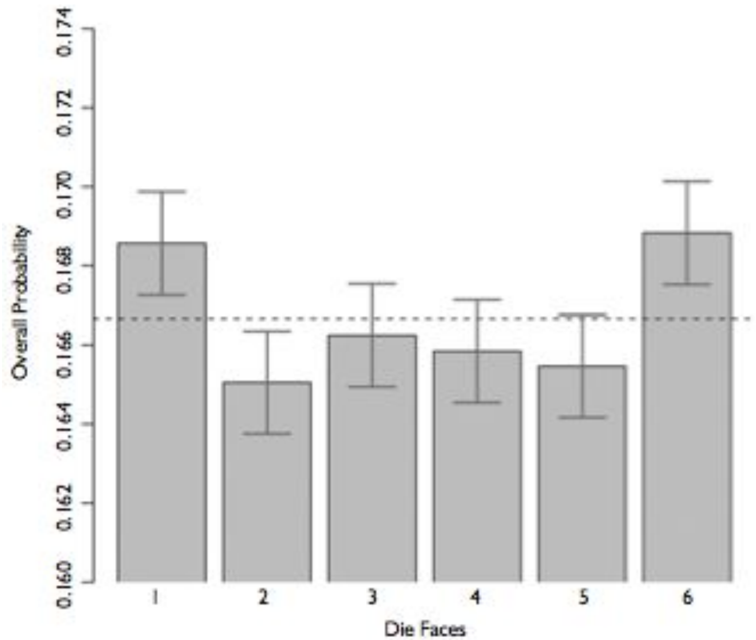
Labby's dice



- In 2009, Zacariah Labby (U of Chicago), repeated Weldon's experiment using a homemade dice-throwing, pip counting machine.
- The rolling-imaging process took about 20 seconds per roll.
- Each day there were ~150 images to process manually.
- At this rate Weldon's experiment was repeated in a little more than six full days.

Labby's dice (cont.)

- Labby did not actually observe the same phenomenon that Weldon observed (higher frequency of 5s and 6s).
- Automation allowed Labby to collect more data than Weldon did in 1894, instead of recording "successes" and "failures", Labby recorded the individual number of pips on each die.



Expected counts

Labby rolled 12 dice 26,306 times. If each side is equally likely to come up, how many 1s, 2s, ..., 6s would he expect to have observed?

(a) 1/6

(b) 12/6

(c) 26,306 / 6

(d) 12 x 26,306 / 6 = 52,612

Summarizing Labby's results

The table below shows the observed and expected counts from Labby's experiment.

Outcome	Observed	Expected
1	53,222	52,612
2	52,118	52,612
3	52,465	52,612
4	52,338	52,612
5	52,244	52,612
6	53,285	52,612
Total	315,672	315,672

Why are the expected counts the same for all outcomes but the observed counts are different? At a first glance, does there appear to be an inconsistency between the observed and expected counts?

Setting the hypotheses

Do these data provide convincing evidence of an inconsistency between the observed and expected counts?

H_0: There is no inconsistency between the observed and the expected counts. The observed counts follow the same distribution as the expected counts.

H_A: There is an inconsistency between the observed and the expected counts. The observed counts *do not* follow the same distribution as the expected counts. There is a bias in which side comes up on the roll of a die.

Evaluating the hypotheses

- To evaluate these hypotheses, we quantify how different the observed counts are from the expected counts.
- Large deviations from what would be expected based on sampling variation (chance) alone provide strong evidence for the alternative hypothesis.
- This is called a *goodness of fit* test since we're evaluating how well the observed data fit the expected distribution.

Anatomy of a test statistic

The general form of a test statistic is

$$\frac{\text{point estimate} - \text{null value}}{\text{SE of point estimate}}$$

This construction is based on

1. identifying the difference between a point estimate and an expected value if the null hypothesis was true, and
2. standardizing that difference using the standard error of the point estimate.

These two ideas will help in the construction of an appropriate test statistic for count data.

Chi-square statistic

When dealing with counts and investigating how far the observed counts are from the expected counts, we use a new test statistic called the *chi-square (χ^2) statistic*.

χ^2 statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O - E)^2}{E} \qquad \text{where } k = \text{total number of cells}$$

Calculating the chi-square statistic

Outcome	Observed	Expected	$\frac{(O-E)^2}{E}$
1	53,222	52,612	$\frac{(53{,}222-52{,}612)^2}{52{,}612} = 7.07$
2	52,118	52,612	$\frac{(52{,}118-52{,}612)^2}{52{,}612} = 4.64$
3	52,465	52,612	$\frac{(52{,}465-52{,}612)^2}{52{,}612} = 0.41$
4	52,338	52,612	$\frac{(52{,}338-52{,}612)^2}{52{,}612} = 1.43$
5	52,244	52,612	$\frac{(52{,}244-52{,}612)^2}{52{,}612} = 2.57$
6	53,285	52,612	$\frac{(53{,}285-52{,}612)^2}{52{,}612} = 8.61$
Total	315,672	315,672	24.73

Why square?

Squaring the difference between the observed and the expected outcome does two things:

- Any standardized difference that is squared will now be positive.
- Differences that already looked unusual will become much larger after being squared.

When have we seen this before?

When we think back to the formula for variance

The chi-square distribution

- In order to determine if the χ^2 statistic we calculated is considered unusually high or not we need to first describe its distribution.
- The chi-square distribution has just one parameter called *degrees of freedom (df)*, which influences the shape, center, and spread of the distribution.

Practice

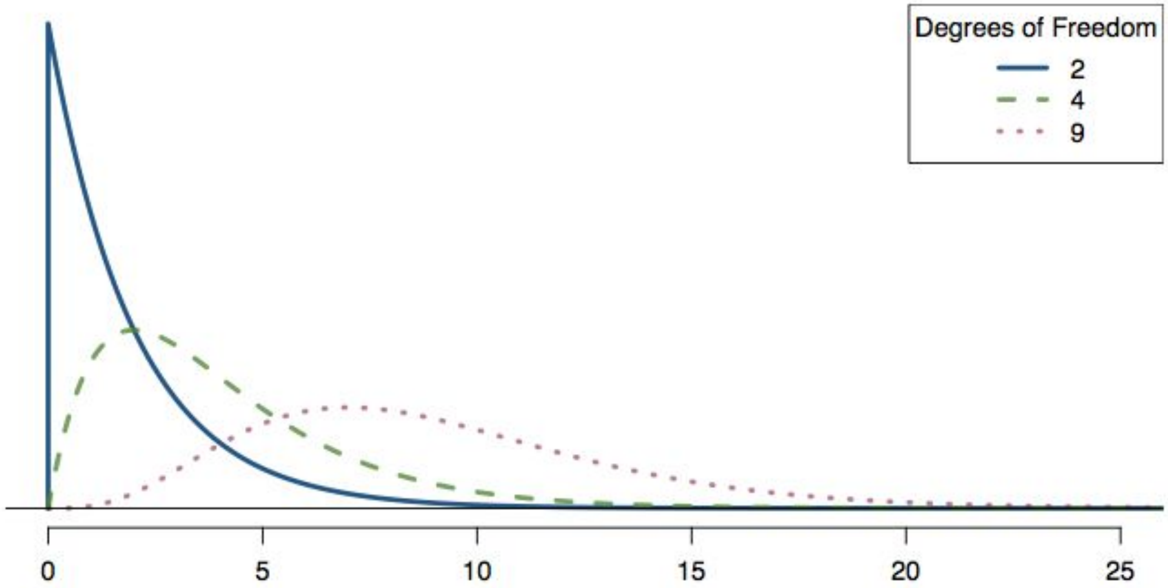
Which of the following is false?

As the df increases,

(a) the center of the χ^2 distribution increases as well

(b) the variability of the χ^2 distribution increases as well

(c) the shape of the χ^2 distribution becomes more skewed (less like a normal)

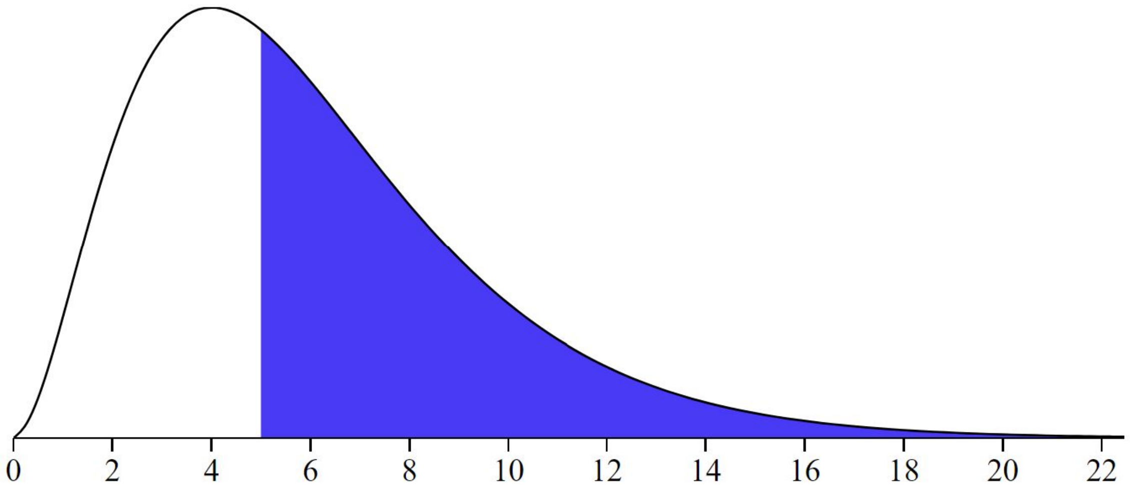


Finding areas under the chi-square curve

- p-value = tail area under the chi-square distribution (as usual)
- We will use the R function pchisq( ) where we specify the test statistic and the degrees of freedom
- Remember that by default we will be given area to the left

Finding areas under the chi-square curve

Calculate the area above a cutoff value of 5 for the chi-square curve with *df* = 6.

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> 1 - pchisq(5, df = 6)
[1] 0.5438131
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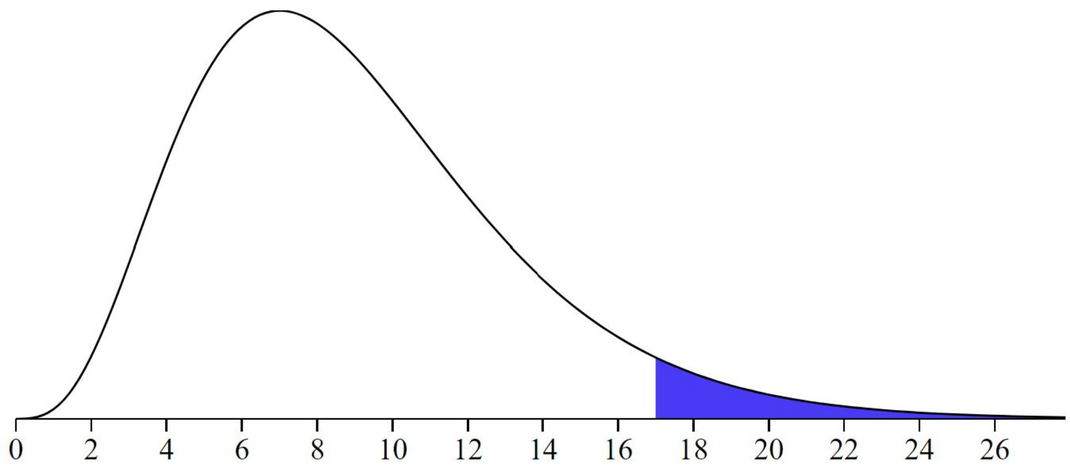
54.38% of the area under the chi-square distribution curve with 6 degrees of freedom is above 5.

Finding areas under the chi-square curve

Calculate the shaded area above a cutoff value of 17 for the chi-square curve with *df = 9*.

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> 1 - pchisq(17, df = 9)
[1] 0.04871598
```

4.87% of the area under the chi-square distribution curve with 9 degrees of freedom is above 17.

Back to Labby's dice

- The research question was: Do these data provide convincing evidence of an inconsistency between the observed and expected counts?
- The hypotheses were:

  H_0: There is no inconsistency between the observed and the expected counts. The observed counts follow the same distribution as the expected counts.

  H_A: There is an inconsistency between the observed and the expected counts. The observed counts *do not* follow the same distribution as the expected counts. There is a bias in which side comes up on the roll of a die.

- We had calculated a test statistic of $\chi^2 = 24.67$.
- All we need is the df and we can calculate the tail area (the p-value) and make a decision on the hypotheses.

Degrees of freedom for a goodness of fit test

- When conducting a goodness of fit test to evaluate how well the observed data follow an expected distribution, the degrees of freedom are calculated as the number of cells/levels (*k*) minus 1.

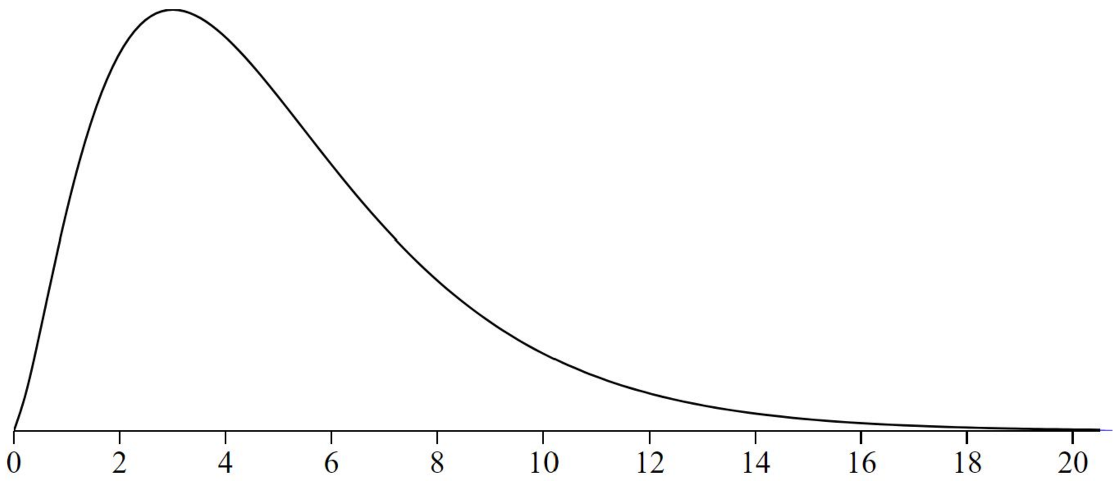
$$df = k - 1$$

- For dice outcomes, k = 6, therefore

$$df = 6 - 1 = 5$$

Finding a p-value for a chi-square test

The *p-value* for a chi-square test is defined as the *tail area above the calculated test statistic.*

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> 1 - pchisq(24.67, df = 5)
[1] 0.001613338
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There is a .02% probability of observing data as or more different, if the faces of the dice are all equally likely.

Conclusion of the hypothesis test

We calculated a p-value 0.0002. At 5% significance level, what is the conclusion of the hypothesis test?

(a) Reject H_0, the data provide evidence to support H_A.
(b) Reject H_0, the data do not provide evidence to support H_A.
(c) Fail to reject H_0, the data provide evidence to support H_A.
(d) Fail to reject H_0, the data do not provide evidence to support H_A.

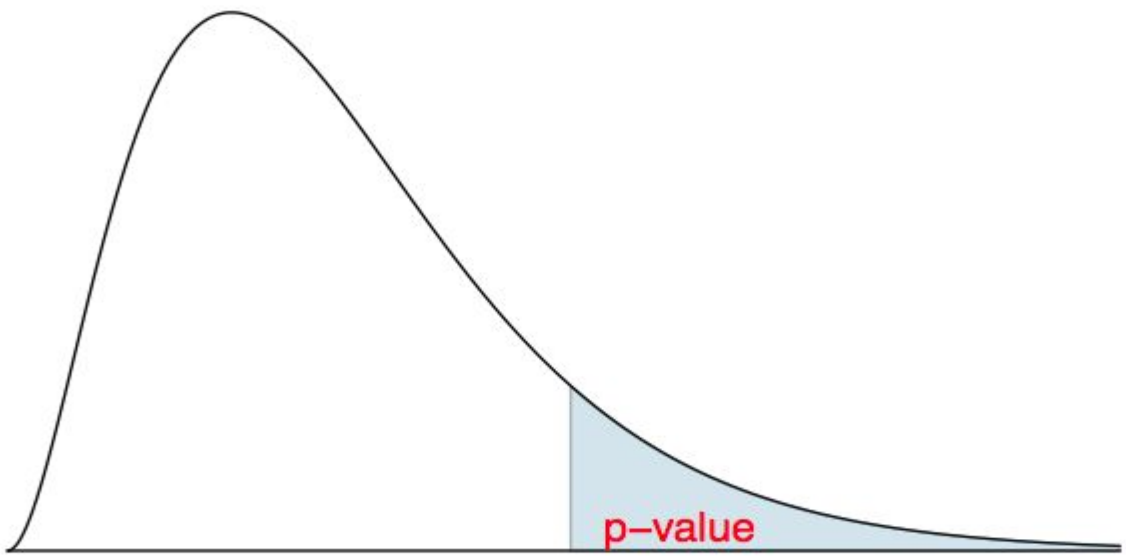
We found strong evidence (p-value = $0.0002 < \alpha = 0.05$) to support the hypothesis that the dice face are bias.

Turns out...

- The 1-6 axis is consistently shorter than the other two (2-5 and 3-4), thereby supporting the hypothesis that the faces with one and six pips are larger than the other faces.
- Pearson's claim that 5s and 6s appear more often due to the carved-out pips is not supported by these data.
- Dice used in casinos have flush faces, where the pips are filled in with a plastic of the same density as the surrounding material and are precisely balanced.

Recap: p-value for a chi-square test

- The p-value for a chi-square test is defined as the tail area *above* the calculated test statistic.
- This is because the test statistic is always positive, and a higher test statistic means a stronger deviation from the null hypothesis.



Conditions for the chi-square test

1. *Independence*: Each case that contributes a count to the table must be independent of all the other cases in the table.
2. *Sample size*: Each particular scenario (i.e. cell) must have at least 5 *expected* cases.
3. *df > 1*: Degrees of freedom must be greater than 1.

Failing to check conditions may unintentionally affect the test's error rates.

2009 Iran Election

There was lots of talk of election fraud in the 2009 Iran election. We'll compare the data from a poll conducted before the election (observed data) to the reported votes in the election to see if the two follow the same distribution.

Candidate	Observed # of voters in poll	Reported % of votes in election
(1) Ahmedinajad	338	63.29%
(2) Mousavi	136	34.10%
(3) Minor candidates	30	2.61%
Total	504	100%
	↓ *observed*	↓ *expected distribution*

Hypotheses

What are the hypotheses for testing if the distributions of reported and polled votes are different?

H_0: The observed counts from the poll follow the same distribution as the reported votes.

H_A: The observed counts from the poll do not follow the same distribution as the reported votes.

Calculation of the test statistic

Candidate	Observed # of voters in poll	Reported % of votes in election	Expected # of votes in poll
(1) Ahmedinajad	338	63.29%	$504 \times 0.6329 = 319$
(2) Mousavi	136	34.10%	$504 \times 0.3410 = 172$
(3) Minor candidates	30	2.61%	$504 \times 0.0261 = 13$
Total	504	100%	504

$$\frac{(O_1 - E_1)^2}{E_1} = \frac{(338 - 319)^2}{319} = 1.13$$

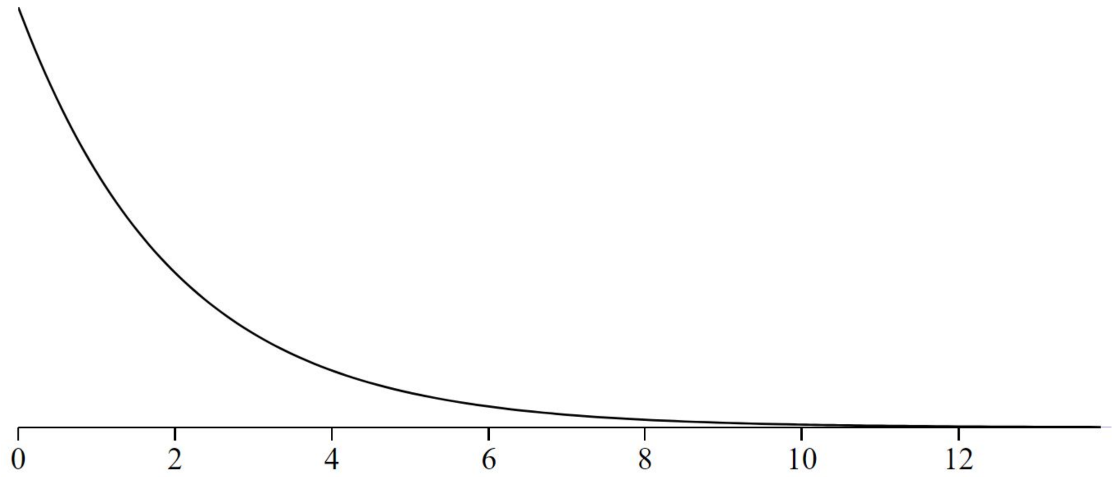
$$\frac{(O_2 - E_2)^2}{E_2} = \frac{(136 - 172)^2}{172} = 7.53$$

$$\frac{(O_2 - E_2)^2}{E_2} = \frac{(30 - 13)^2}{13} = 22.23$$

$$\chi^2_{df=3-1=2} = 30.89$$

Calculation of the p-value

Our test statistic is 30.89 with 2 degrees of freedom. We want to calculate the are above this to get our p-value.

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> 1 - pchisq(30.89, df = 2)
[1] 1.960296e-07
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There is essentially a 0% probability of observing data as or more different from the expected distribution as what we observed, if the data truly follow the expected distribution.

Conclusion

Based on these calculations what is the conclusion of the hypothesis test?

(a) p-value is low, H_0 is rejected. The observed counts from the poll do not follow the same distribution as the reported votes.

(b) p-value is high, H_0 is not rejected. The observed counts from the poll follow the same distribution as the reported votes.

(c) p-value is low, H_0 is rejected. The observed counts from the poll follow the same distribution as the reported votes

(d) p-value is low, H_0 is not rejected. The observed counts from the poll do *not* follow the same distribution as the reported votes.